TMSP 2021/22
Problems 5 (11.04.2022)

1 Functional derivative

- Consider the action $S[\varphi] = \int dt \int dx\, \mathcal{L}(\varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x})$, where $\mathcal{L}(\varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x})$ is the Lagrangian density and $\varphi = \varphi(x,t)$. Write down the equation $\frac{\delta S[\varphi]}{\delta \varphi(x',t')} = 0$

- Consider the covariant formulation of Maxwell electrodynamics in vacuum, where $x^\alpha = (c\,t, \vec{r}) = (ct, x, y, z)$, $A^\alpha = (\varphi/c, \vec{A})$, $j^\alpha = (c\rho, \vec{j})$, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$, $F^{\mu\nu} = \eta^{\mu\alpha}\, F_{\alpha\beta}\, \eta^{\beta\nu}$ (note the summation over repeated indices), and

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

  The lagrangian density has the form $\mathcal{L} = -\frac{1}{4\mu_0}\, F^{\alpha\beta}\, F_{\alpha\beta} - A_\alpha\, j^\alpha$, where μ_0 is the permeability of vacuum. Check that the corresponding Euler-Lagrange equations obtained by calculating the functional derivative of the action take the form

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} - \partial_\beta\, \frac{\partial \mathcal{L}}{\partial(\partial_\beta\, A_\alpha)} = 0$$

  and that they correspond to the Gauss and Ampère laws $\partial_\beta\, F^{\beta\alpha} = \mu_0\, j^\alpha$.

2 Capillary Wave Hamiltonian

Consider an interface separating two coexisting phases, say liquid and gas. This interface fluctuates and, in general, is not planar. The position of the interface describing its deviation from the planar configuration (corresponding to the (x, y) - plane) is described by $z = f(x, y)$. The corresponding cost of interfacial undulation (for three dimensional system with gravitational field acting along the z-direction and for small undulations) is described by the so-called capillary wave Hamiltonian

$$\mathcal{H}_{CW} = \frac{1}{2} \underbrace{\int dx \int dy}_{A} \Big[\sigma_{\ell g} (\nabla f(x,y))^2 + g(\rho_\ell - \rho_g) f^2(x,y) \Big] ,$$

where $A = L^2$ denotes the area of the interface in the planar case and $\sigma_{\ell g}$ is the coefficients of surface tension.
In the d-dimensional case the above expression can be generalized to

$$\mathcal{H}_{cw}[f] = \frac{1}{2} \int d\vec{R} \, \Big[\sigma_{\ell g} \, (\nabla f(\vec{R}))^2 + g \, \Delta\rho \, f^2(\vec{R}) \Big]$$

where $\vec{R} = (x_1, \cdots, x_{d-1})$. Discuss the interfacial width $\left\langle f^2(\vec{R}) \right\rangle$ as function of system size L and g in $d = 2$ and $d = 4$. In each case consider two limiting procedures: $L \to \infty$ at $g \neq 0$, and $g \to 0$ at finite L.